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On the Principle of Exchange of Stabilities of Thermohaline Problem in Porous Medium with Variable Gravity using Positive Operator

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ABSTRACT: In the present paper, Thermohaline Problem with variable gravity in porous medium is analyzed and it is established by the method of positive operator of Weinberger and uses the positivity properties of Green's function that principle of exchange of stabilities is valid for this general problem, when g(z) is non-negative throughout the fluid layer and $\Re \geq \Re_s$.

Keywords: Thermohaline problem; variable gravity and principle of exchange of stabilities.

INTRODUCTION: In classical thermal instability problems, it has been assumed that the driving density differences are produced by the spatial variation of single diffusing property i.e. heat. Recently, it has been shown that a new phenomenon occurs when the simultaneous presence of two or more components with different diffusivities is considered. This problem has been probed when we think about ocean where both heat and salt (or some other dissolved substances) are important. This problem has been termed as thermosolutal convection (or thermohaline convection*)*. In these problems the solute is commonly, but not necessarily, a salt. Related effects have now been observed in other contexts, and the name 'double–diffusive convection', has been used to cover this wide range of phenomena. For a broad view of the subject one may be referred to Brandt and Fernando [1]. Two fundamental configurations have been studied in the context of the thermohaline instability problem, one by Stern [11], wherein the temperature gradient is stabilizing and the concentration gradient is destabilizing, and another by Veronis [14], wherein the temperature gradient is destabilizing and the concentration gradient is stabilizing. The main results derived by Stern and Veronis for their respective configurations are that both allow the occurrence of a steady motion or an oscillatory motion of growing amplitude provided the destabilizing concentration gradient or the temperature gradient is sufficiently large. However, steady motion is the preferred mode of onset of instability in the case of Stern's configuration, whereas oscillatory motions of growing amplitude are preferred in Veronis's configuration. Furthermore, these results are independent of the initially gravitationally stable or unstable character of the two configurations. It is important to note here that Veronis's work is restricted to dynamically free boundaries, whereas Stern's work assumes the "principle of exchange of stabilities." Keeping in view the foregoing discussion, thermohaline configurations of Veronis and Stern types can therefore be further classified into the following two classes: (i) the first class, in which thermohaline instability manifests itself when the total density field is initially bottom heavy, and (ii) the second class, in which thermohaline instability manifests itself when the total density field is initially top heavy. aries at fixed concentrations, and since then many others [6]. The problem of thermosolutal convection in a layer of fluid heated from below and subjected to a stable solute gradient has been studied by Veronis [14]. The minimum requirements for the occurrence of thermosolutal convection are the following:

i) The fluid must contain two or more components having different molecular diffusivities. It is the differential diffusion that produces the density differences required to derive the motion.

ii) The components must make opposing contributions to the vertical density gradient.Further, he found that the analogous non–dimensional parameters accounting for uniform salinity gradient are given by $S = \frac{g\alpha'\beta' d^4}{\upsilon\kappa'_T}$ and Schmidt number $q = \frac{\upsilon}{\kappa'_T}$, where α', β', κ'_T denote the coefficient of analogous solvent expansion, uniform solute gradient and solute diffusivity, respectively. The main large–scale engineering applications of double–diffusive concepts are to solar ponds, shallow artificial lakes that are density stratified. Linear calculations have also been made for a variety of boundary conditions by Nield [7] and for an unbounded fluid by Walin [15]. A study of the onset of convection in a layer of sugar solution, with a stabilizing concentration gradient, when the layer is heated from below, has been made by Shirtcliff [7]. He found that the first stage of the development of convection layers similar to those described by Turner and Stommel [15]. Nield [9] has studied the problem of thermohaline convection in a horizontal layer of viscous fluid heated from below and salted from above. When the solute gradient is stabilizing, Sani [14] has found that finite amplitude subcritical instability (convection at a thermal Rayleigh number less than that given by the linear theory) is possible. A direct analogue of heat/salt diffusive convection has been used to explain the properties of large stars with helium–rich core, which is heated from below and thus convecting. Spiegel [10] has shown that variations in the helium/hydrogen ratio can produce a density gradient that limits the helium transport by double–diffusive convection, though, whether this may be in layers is still unclear. Another example of double–diffusive convection is when metals solidify, since as metals solidify, undesirable inhomogeneities on the microscopic scale can be produced by several mechanisms, among which is double–diffusive convection. It was shown by Turner [12,13] that the form of the resulting motions depends on whether the deriving energy comes from the component having the higher or lower diffusivity. When one layer of fluid is placed above another (denser) layer having different diffusive properties, two basic types of convective instabilities arise, in the 'diffusive' and 'finger' configurations. In both the cases, the double–diffusive fluxes can be much larger than the vertical transport in a single–component fluid because of the coupling between diffusive and convective processes.The salinity gradient is not constant with depth and this has prompted theoretical studies [16] of the breakdown, which is found to occur preferentially (in agreement with observations) in a thin layer where the salinity gradient is a minimum. A recent comprehensive review of thermosolutal convection in porous media has been conducted by Nield and Bejan [8]. For a broad and latest review of the subject one may be referred to Turner and Brandt and Fernando [1] and Gupta *et al*. [3,4]. Dhiman, *et al*. [2] have also dealt with the problem On the Stationary Convection of Thermohaline Problems of Veronis and Stern Types. In the present paper, Thermohaline Problem in porous medium is heated from below with variable gravity is analyzed and it is established by the method of positive operator.

METHODS:

Mathematical Formulation of the Physical Problem: Basic Hydrodynamical Equations Governing the Physical Configuration.

The basic hydrodynamic equations that govern the above physical configurations under Boussinesq approximation for the present problem are given by (c.f. Veronis [14]);

$$\left(D^2-k^2\right)\left(\tfrac{\sigma}{P_r}+\frac{1}{p}\right)w = g\left(z\right)R_T k^2\theta - g(z)R_s k^2\varphi \qquad (1)$$

$$\left(D^2-k^2-A\sigma\right)\theta = -R_T w \qquad (2)$$

$$\left(D^2-k^2-\frac{\sigma}{\tau}\right)\varphi = -\frac{R_s}{\tau}w \qquad (3)$$

together with following dynamically free and thermally and electrically perfectly conducting boundary conditions

$$w = 0 = \theta = D^2 w \quad a\, z = 0 \; and \; z = 1 \qquad (4)$$

where, σ is the complex constant, $\Re = R^2{}_T = \frac{g_0\alpha\beta d^4}{K_T\upsilon}$ is the thermal Rayleigh number, Pr= $\frac{\upsilon}{\kappa}$ is the Prandtl number, $\Re_s = R^2{}_s = \frac{g_0\alpha'\beta' d^4}{K_T\upsilon}$ is the salinity

Rayleigh number, $\tau = \dfrac{K_S}{K_T}$ is the Lewis number, $A = \dfrac{\rho_{s_0} c_{s_0}}{\rho_0 c_0}$ and $p = \dfrac{k_1}{\varepsilon d^2}$ are two positive constants.

Method of Positive Operator: We seek conditions under which solutions of equations (1)-(3) together with the boundary conditions (4) grow. To apply the method of positive operator, formulate the above equations (1) - (3) together with boundary conditions (4) in terms of certain operators as;

$$\left(\frac{\sigma}{P_r} + \frac{1}{p}\right)\tilde{M}w = g(z) R_T k^2 \theta - g(z) R_s k^2 \varphi \tag{5}$$

$$\left(\tilde{M} + A\sigma\right)\theta \quad = \quad - R_T w \tag{6}$$

$$\left(\tilde{M} + \frac{\sigma}{\tau}\right)\phi \quad - \frac{R_s}{\tau} w \tag{7}$$

The domains are contained in B, where $B = L^2(0,1) = \left\{ \varphi \;\middle|\; \int_0^1 |\varphi|^2 \, dz < \infty \right\}$,

with scalar product $\langle \phi, \varphi \rangle = \int_0^1 \phi(z)\, \overline{\varphi(z)}\, dz$, $\phi, \varphi \in$ B ; and norm $\|\phi\| = \langle \phi, \phi \rangle^{1/2}$.

We know that $\boldsymbol{L}^2$ $(\mathbf{0}, \mathbf{1})$ is a Hilbert space, so, the domain of M is dom M = $\{\phi \in B / \; D\phi, m\phi \in B, \;\; \phi(0) = \phi(1) = 0\}$.

We can formulate the homogeneous problem corresponding to equations (1)-(3) by eliminating θ from (5) - (7) as;

$$w = k^2 \tilde{M}^{-1} \left(\frac{1}{p} + \frac{\sigma}{p_r}\right)^{-1} g(z) \left(\Re(\tilde{M} + A\sigma)^{-1} - \frac{\Re_S}{\tau}(\tilde{M} + \frac{\sigma}{\tau})^{-1}\right) w \tag{8}$$

$$w = K(\sigma) w$$

where, $k(\sigma) = k^2 \tilde{M}^{-1} \left(\dfrac{1}{p} + \dfrac{\sigma}{p_r}\right)^{-1} g(z) \left(\Re(\tilde{M} + A\sigma)^{-1} - \dfrac{\Re_S}{\tau}(\tilde{M} + \dfrac{\sigma}{\tau})^{-1}\right)$ (9)

is the linearized stability operator. Further $\Re$ and $\Re_s$ respectively are the thermal and concentration Rayleigh numbers. In the present problem the linearized stability operator $K(\sigma)$ consists of three different operators, namely $\tilde{M}^{-1}, \left(\dfrac{1}{p} + \dfrac{\sigma}{p_r}\right)^{-1}$ *and* $\left(\Re(\tilde{M} + A\sigma)^{-1} - \dfrac{\Re_S}{\tau}(\tilde{M} + \dfrac{\sigma}{\tau})^{-1}\right)$, however the operator $\left(\Re(\tilde{M} + A\sigma)^{-1} - \dfrac{\Re_S}{\tau}(\tilde{M} + \dfrac{\sigma}{\tau})^{-1}\right)$ which is the difference of two operators $\Re\left(\tilde{M} + A\sigma\right)^{-1}$ *and* $\dfrac{\Re s}{\tau}(M + \dfrac{\sigma}{\tau})^{-1}$ needs special attention in regards to the positivity of the operator.

Defining, $T\left(\dfrac{\sigma}{p_r}\right)$ *exists for* $\sigma \in T_{k\sqrt{\Pr}} = \{\sigma \in C \;|\operatorname{Re}(\sigma) > -k^2 \Pr, \operatorname{Im}(\sigma) = 0\}$

and $\left\|T\left(\frac{\sigma}{\Pr}\right)\right\|^{-1} > \left|\sigma + k^2 \Pr\right| \; for \operatorname{Re}(\sigma) > -k^2 \Pr$. and define $T\left(\frac{\sigma}{\tau}\right)$ by $T\left(\frac{\sigma}{\tau}\right) f = \int_0^1 g\left(z, \xi; \frac{\sigma}{\tau}\right) f(\xi) d\xi$, where g $\left(z, \xi, \frac{\sigma}{\tau}\right)$ is Green's function kernel for the operator $\left(\tilde{M} + \frac{\sigma}{\tau}\right)^{-1}$.

Let, $L(\sigma)$ = $\left(\Re(\tilde{M} + A\sigma)^{-1} - \frac{\Re_S}{\tau}(\tilde{M} + \frac{\sigma}{\tau})^{-1}\right)$. The operator $L(\sigma)$ exists for $\sigma \in L_k = \left\{\sigma \in C \; \middle| \operatorname{Re}(\sigma) > \max.\{-k^2(1,\tau)\}, \operatorname{Im}(\sigma) = 0\right\}$ and $\|L(\sigma)\|^{-1} > \left|\sigma + k^2\right|$ for $\left\{\operatorname{Re}(\sigma) > \max\{-k^2(1,\tau)\}, \operatorname{Im}(\sigma) = 0\right\}$. L (σ) = $[\Re T(\sigma) - \frac{\Re_s}{\tau} T(\frac{\sigma}{\tau})]$ is an integral operator with Green's function kernel

$$g'(z,\xi,\sigma) = \frac{\Re \; \cosh[r(1-|z-\xi|)] - \Re \cosh[r(-1+z+\xi)]}{2r \sinh r} - \frac{\frac{\Re_s}{\tau}\cosh[r'(1-|z-\xi|)] - \frac{\Re_s}{\tau}\cosh[r'(-1+z+\xi)]}{2r' \sinh r'}$$

for $\sigma > \max.\{-k^2(1,\tau)\}$.

$K(\sigma)$ defined in (9), which is a composition of certain integral operators, is termed as linearized stability operator. K (σ) depends analytically on σ in a certain right half of the complex plane. It is clear from the composition of K (σ) that it contain an implicit function of σ .

We shall examine the resolvent of the K (σ) defined as $[I - K(\sigma)]^{-1}$

$$[I - K(\sigma)]^{-1} = \left\{I - [I - K(\sigma_0)]^{-1}[K(\sigma) - K(\sigma_0)]\right\}^{-1}[I - K(\sigma_0)]^{-1} \tag{10}$$

If for all σ_0 greater than some a,

Remarks:

(1) $[I - K(\sigma_0)]^{-1}$ is positive,

(2) $K(\sigma)$ has a power series about σ_0 in $(\sigma_0 - \sigma)$ with positive coefficients; i.e., $\left(-\frac{d}{d\sigma}\right)^n K(\sigma_o)$ is positive for all n, then the right side of (10) has an expansion in $(\sigma_0 - \sigma)$ with positive coefficients. Hence, we may apply the methods of Weinberger [1969] and Rabinowitz , to show that there

exists a real eigenvalue σ_1 such that the spectrum of $K(\sigma)$ lies in the set $\{\sigma : \operatorname{Re}(\sigma) \le \sigma_1\}$. This is result is equivalent to PES, which was stated earlier as "the first unstable eigenvalue of the linearized system has imaginary part equal to zero."

RESULTS AND DISCUSSION:

The Principle of Exchange of Stabilities (PES):

It is clear that $K(\sigma)$ is a product of certain operators. Condition (1) can be easily verified by following the analysis of Herron [3,5] for the present operator $K(\sigma)$, i.e. $K(\sigma)$ is a linear, compact integral operator, and has a power series about σ_0 in $(\sigma_0 - \sigma)$ with positive coefficients. Thus, $K(\sigma)$ is a positive operator leaving invariant a cone (set of non negative functions). Moreover, for σ real and sufficiently large, the norms of the operators $T(0)$ *and* $T(\Pr\sigma)$

become arbitrarily small. So, $\|K(\sigma)\| < 1$. Hence, $[I - K(\sigma)]^{-1}$ has a convergent Neumann series, which implies that $[I - K(\sigma)]^{-1}$ is a positive operator. This is the content of condition (P1).

To verify condition (2), we note that. Green's function kernel g $\left(z, \xi, \frac{\sigma}{\Pr}\right)$ is the Laplace transform of the Green's function PrG$(z, \xi; \Pr t)$ for the initial-boundary value problem

$$\left(-\frac{\partial^2}{\partial z^2} + k^2 + \frac{1}{\Pr}\frac{\partial}{\partial t}\right)G = \delta(z - \xi, t),$$

where, $\delta(z - \xi, t)$ is Dirac –delta function in two-dimension,

with boundary conditions

$$G(0, \xi; \Pr t) = G(1, \xi; \Pr t) = G(z, \xi; 0) = 0,$$

then $G(z, \xi; \Pr t) \geq 0$.

and Green's function kernel $g'(z, \xi; \sigma)$ is the Laplace transform of the Green's function G' (z, , ξ, t) defined by

G' (z, , ξ, t) = $\Re$ G $(z, \xi; t)$ – $\Re_s$ $G(z, \xi; \pi t)$ If $\Re \geq \Re_s$ .then G' (z, , ξ, t) $\geq$ 0

With boundary conditions

$$G(0, \xi; t) = G(1, \xi; t) = G(z, \xi; 0) = 0,$$

Using the similar result proved in Herron [3] by direct calculation of the inverse Laplace transform, we have $K(\sigma)$ is a positive operator for all real $\sigma_0 > \max.\{-k^2(1, \Pr, \tau)\}$ and $\Re \geq \Re_s$ together with g(z) positive in the flow domain.

Theorem: The PES holds for (12) - (14) when g(z) is nonnegative throughout the fluid domain and $\sigma_0 > \max.\{-k^2(1, \Pr, \tau)\}$ and $\Re \geq \Re_s$.

Proof: As $[I - K(\sigma)]$ is a nonnegative compact integral operator for $\sigma_0 > \max.\{-k^2(1, \Pr, \tau)\}$ and for $\Re \geq \Re_s$ which satisfied all the conditions of the Krein-Rutman theorem and hence it has a positive eigen value σ_1, which is an upper bound for the absolute values of all the eigenvalues, and the corresponding eigen function $\phi(\sigma)$ is nonnegative, which is essentially the contents of condition (2) stated in Remark 1.

We observe that

$$[I - K(\sigma)][\phi(\sigma)] = (1 - \sigma_1)\phi \geq 0,$$

Thus, if $[I - K'''(\sigma)]$ is nonnegative, then $\sigma_1 \leq 1$. The methods of Weinberger and abinowitz [1969] apply thereby showing that there exits a real eigenvalue $\sigma_1 \leq 1$ such that the spectrum of $K(\sigma)$ lies in the set $\{\sigma \mid \operatorname{Re}(\sigma) \leq \sigma_1\}$.

This is equivalent to the PES.

CONCLUSIONS: In this paper we have investigated the Thermohaline Convection Problem with variable gravity in porous medium. It is established that if g(z) is positive; throughout the flow domain, then PES is valid for Thermohaline Convection if $\Re \geq \Re_s$.

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