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Derivation of Equations of Phase Dynamics in a Stack of Long Josephson Junctions with Multi-gap Superconductors

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ABSTRACT: In the present work, we study the phase dynamics of long Josephson junction (LJJ) with stack of multi-gap superconductors like iron pnictides, MgB_2, etc. This is an extension of the Ambegaoker-Baratoff relation for a Josephson junction of single gap superconductor to the stack of multi-gap junctions. The starting point of our derivation is to develop the quantum mechanical Hamiltonian of the system and then to write the corresponding partition function. The partition function is further simplified through the phenomenological procedure followed by Hubbard-Stratonovich transformation, Grassmann integration, and saddle-point approximation. We then obtain the action functional which is further simplified using Goldston mode. Finally the equation for phase dynamics can be derived using Euler-Lagrange equation of motion. Our generalized theoretical result has been compared to other's for two-gap junctions and found to be close agreement to each other.

Keywords: Model Hamiltonian; action functional; Hubbard-Stratonovich transformation and Goldston mode.

INTRODUCTION: Superconductivity was discovered in 1911 by H. Kamarlingh Onnes [1] in Leiden, just 3 year after he had first liquefied helium. In 1986, a new class of high temperature superconductor was discovered by Bednorz and Mueler [2]. The perfect diamagnetic characteristics of the superconductor was discovered by Meissner and Ochsenfeld [3] in 1933. In 1935, the brothers F. and H. London [4] proposed the two electrodynamics equations to govern the microscopic electric and magnetic field. Pippard [5] introduced the coherence length while proposing a nonlocal generalization of the London equations.

The next step in the evolution of superconductor was the establishment of the existence of the energy gap Δ, of the order $k_B T_c$ between the ground state and quasi-particle excitation of the system by Daunt and Mendelssohn [6]. In 1957, Bardeen, Cooper and Schrieffer [7] propounded a pairing theory of superconductivity, which was named as BCS theory, according to which electron-phonon interaction causes an instability of the ordinary Fermi-sea ground state of the electron gas with respect to the formation of bound pairs electrons occupying sate with equal and opposite momentum and spin. These pairs are called Cooper pairs and behave as boson particles. Josephson [8] predicted that Cooper pairs should be able to tunnel between two superconductors even at zero voltage difference giving a super-current density. This is called the Josephson effect and is one of the most important and drastic phenomenon in superconductivity [9]. Many experimental and theoretical researches have been done on tunneling between one-gap superconductors [10–12], but the Josephson effect in the multi-gap superconductors is the quite interesting due to presence of some important phenomena such as collective oscillation of fluxons, time reversal symmetry (TRS) breaking, emission of THz frequencies etc. [13–16]. Multi-gap superconductors have been observed since the discovery of iron-based high T_c superconductors [17,18]. In this type of superconductor, 3rd electrons of the iron atom form multi-bands whose Cooper pairs condense into a multi-gap superconducting state [15]. The phase difference may be understood in term of the interplay between the inter-band and intra-band Josephson effects. Inter-band Josephson effect describes tunneling between the two electronic bands in each layer of superconductors whereas intra-band Josephson Effect describes tunneling between two adjacent superconducting layers. These effects can be reflected into the dynamics of the phase difference between the condensates within the same superconducting layer and across two adjacent superconducting layers, respectively [14]. Recently, various types of Josephson junctions with iron-based superconductors have been fabricated and typical Josephson effects have been confirmed.

The present paper deals about the development of equation of phase dynamics of fluxons while flowing at the junction under the application of external magnetic field with appropriate biasing voltage. This will discuss in details in the following section. The obtained equations will be compared to those developed by others for some special cases such as two-gap junctions. The work will be ended by drawing some conclusions.

Theoretical Development:

Defining the model Hamiltonian of the system: The starting point of the present work is to write total Hamiltonian of the system which comprises the free Hamiltonian (H_{free}), pairing Hamiltonian (H_{pair}) and tunneling Hamiltonian (H_T) i.e.

$$H = H_{\text{free}} + H_{\text{pair}} + H_T \tag{1}$$

Each site of a superconducting system consists of formions with spin up $(\uparrow)$ and spins down $(\downarrow)$. The total free (non-interacting) Hamiltonian of this site is defined as

$$H_{\text{free}} = \sum_{l,i,\sigma}\int d^3 r C^{\dagger}_{l,i,\sigma}\left[\frac{1}{2m}(i\hbar\nabla + e^*\vec{A}_l)^2 + e^* A^0_l\right]C_{l,i,\sigma} \tag{2}$$

Here, $C^{\dagger}_{l,i,\sigma}(C_{l,i,\sigma})$ is the creation(annihilation) operator for fermion with spin $\sigma = (\uparrow \text{ or } \downarrow)$ at l-layer and $i-$band. These operators are the function of spatial coordinate $\vec{r}$ and imaginary time $\tau = -it$. $C^{\dagger}_{l,i,\sigma}$ creates a fermion with spin σ at the given site $(\vec{r},\tau)$ and $C_{l,i,\sigma}$ destroy the fermion from there. $C^{\dagger}_{\sigma}$ and C_{σ} have the dimension of inverse square root of volume (i.e. $\Omega^{-1/2}$), with Ω as the total volume of the system. $\vec{A}_l$ and A^0_l are the magnetic vector potential and electric scalar potential respectively. $e^* = 2e$ and e is the electronic charge and m is the mass of a fermion. The operator $-i\hbar\nabla - e^*\vec{A}_l$ is called the canonical momentum operator.

The pairing of any two fermions with opposite spins is possible due to short or long range phonon mediated attractive coupling. After the pairing process, the fermionic nature of particle will destroy and the new bosonic particle forms which is called the Cooper pair. The Hamiltonian associated for this is

$$H_{\text{pair}} = \sum_{l,l',i,i'}\int d^3 r V^{i,i'}_{l,l'} C^{\dagger}_{l,i,\uparrow} C^{\dagger}_{l,i,\downarrow} C_{l',i',\downarrow} C_{l',i',\uparrow} \tag{3}$$

For $l = l'$ and $i = i'$, pairing is intra-layer and intra-band, for $l \neq l'$ and $i = i'$, the pairing is inter-layer and intra-band, for $l = l'$ and $i \neq i'$, the pairing is intra-layer and inter-band, for $l \neq l'$ and $i \neq i'$, the pairing is inter-layer and inter-band.

Similarly, the Hamiltonian associated for tunneling is

$$H_T = \sum_{l,i,i',\sigma}\int d^3 r\left[T^{i,i'}_{l,l+1} C^{\dagger}_{l,i,\sigma} C_{l+1,i',\sigma} + T^{*i',i}_{l+1,l} C^{\dagger}_{l+1,i',\sigma} C_{l,i,\sigma}\right] \tag{4}$$

The first term of Equation (4) infers that the a fermion of spin σ destroys in the i'^{th} band of $l+1^{\text{th}}$ layer and creates in the i^{th} band of l^{th} layer and the second term infers vice-versa of this. $T^{i,i'}_{l,l+1}$ is the tunnel matrix element.

Action functional and path-integral formalism

The action functional is defined as

$$S = \int L dt \tag{5}$$

where L is the Lagrangian given as [19–21]

$$L = \int d^3 r \mathrm{L} \tag{6}$$

with L as the Lagrangian density.

In term of total Hamiltonian, the action function is defined as [16]

$$S = \int_0^{\hbar\beta} d\tau\left[\left(\int d^3 r \sum_{l,i,\sigma} C^{\dagger}_{l,i,\sigma} i\hbar\frac{\partial}{\partial t} C_{l,i,\sigma}\right) + H - \mu N\right] \tag{7}$$

Here, μ is the chemical potential, and N is the total particle number, $\beta = \dfrac{1}{k_B T}$, where k_B is the Boltzmann constant and T is absolute temperature. μN is given as

$$\mu N = \sum_{l,i,\sigma}\int d^3 r \mu_\sigma C^{\dagger}_{l,i,\sigma} C_{l,i,\sigma} \tag{8}$$

Using equations (1), (7) and (8), we get the action functional as

$$S=\int_0^{\hbar\beta}d\tau\left[\left(\int d^3r\sum_{l,i,\sigma}C^\dagger_{l,i,\sigma}(\hbar\frac{\partial}{\partial\tau}-\mu_\sigma)C_{l,i,\sigma}\right)+H_{\text{free}}+H_{\text{pair}}+H_T\right] \tag{9}$$

Substituting the general expressions for H_{free}, H_{pair} and H_T in above equation, the action functional becomes

$$\begin{aligned}S=&\underbrace{\int_0^{\hbar\beta}d\tau\int d^3r\sum_{l,i,\sigma}C^\dagger_{l,i,\sigma}\left(\hbar\frac{\partial}{\partial\tau}+\frac{1}{2m}(i\hbar\nabla+e^*\vec{A}_l)^2+e^*A_l^0-\mu_\sigma)\right)C_{l,i,\sigma}}_{S_{\text{free}}}\\&+\underbrace{\int_0^{\hbar\beta}d\tau\int dr^3\sum_{l,l',i,i'}V^{i,i'}_{l,l'}C^\dagger_{l,i,\uparrow}C^\dagger_{l,i,\downarrow}C_{l',i',\downarrow}C_{l',i',\uparrow}}_{S_{\text{pair}}}+\underbrace{\int_0^{\hbar\beta}d\tau\int d^3r\sum_{l,i,i',\sigma}\left[T^{i,i'}_{l,l+1}C^\dagger_{l,i,\sigma}C_{l+1,i',\sigma}+T^{*i',i}_{l+1,l}C^\dagger_{l+1,i',\sigma}C_{l,i,\sigma}\right]}_{S_T}\end{aligned} \tag{10}$$

Now the partition function of the system is

$$Z=\int\mathrm{D}[C^\dagger,C]\exp\left(-\frac{S}{\hbar}\right) \tag{11}$$

Here, C is a column vector with elements $C_{l,i,\sigma}$ and $C^\dagger$ is a row vector with elements $C^\dagger_{l,i,\sigma}$ and $\int\mathrm{D}[C^\dagger,C]$ represents the product of all integrals over the elements of $C^\dagger$ and C.

Hubbard-Stratonovich transformation: The action functional associated to the pair Hamiltonian is in quartic form of four fermionic fields. The partition function of equation (11) can be rewritten as (applying the transformation $\hbar\frac{\partial}{\partial\tau}\pm e^*A_l^0\rightarrow\hbar\frac{\partial}{\partial\tau}$ and $i\hbar\nabla\pm e^*\vec{A}_l\rightarrow i\hbar\nabla$, since $\vec{A}_l$ and A_l^0 are invariant under gauge transformation)

$$\begin{aligned}Z=\int\mathrm{D}[C^\dagger,C]\exp\Big\{-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\int d^3r\Big[&\sum_{l,i,\sigma}C^\dagger_{l,i,\sigma}\left(\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\nabla^2-\mu_\sigma\right)C_{l,i,\sigma}\\&-\sum_{l,l',i,i'}V^{i,i'}_{l,l'}C^\dagger_{l,i,\uparrow}C^\dagger_{l,i,\downarrow}C_{l',i',\downarrow}C_{l',i',\uparrow}+\sum_{l,i,i',\sigma}\left(T^{i,i'}_{l,l+1}C^\dagger_{l,i,\sigma}C_{l+1,i',\sigma}+T^{*i',i}_{l+1,l}C^\dagger_{l+1,i',\sigma}C_{l,i,\sigma}\right)\Big]\Big\}\end{aligned} \tag{12}$$

The quartic terms of fermionic fields can be reduced by using Hubbard-Stratonovich transformation [22,23] and the partition function takes the form

$$\begin{aligned}Z=\int\mathrm{D}[\bar{\Delta},\Delta]\int\mathrm{D}[C^\dagger,C]\exp\Big\{-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\int d^3r\Big[&\sum_{l,i,\sigma}C^\dagger_{l,i,\sigma}\left(\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\nabla^2-\mu_\sigma\right)C_{l,i,\sigma}\\&+\sum_{l,l',i,i'}\left(\bar{\Delta}_{l,i}(V^{-1})^{i,i'}_{l,l'}\Delta_{l',i'}+\bar{\Delta}_{l,i}C_{l,i,\downarrow}C_{l,i,\uparrow}+\Delta_{l,i}C^\dagger_{l,i,\uparrow}C^\dagger_{l,i,\downarrow}\right)+\sum_{l,i,i',\sigma}\left(T^{i,i'}_{l,l+1}C^\dagger_{l,i,\sigma}C_{l+1,i',\sigma}+T^{*i',i}_{l+1,l}C^\dagger_{l+1,i',\sigma}C_{l,i,\sigma}\right)\Big]\Big\}\end{aligned} \tag{13}$$

Here, $\bar{\Delta}(\Delta)$ is the new fields which are bosonic in nature. $\bar{\Delta}$ is a row vector containing the elements $\bar{\Delta}_{l,i}(\vec{r},\tau)$ and Δ is a column vector containing the elements $\Delta_{l,i}(\vec{r},\tau)$. This step is called bosonization.

Nambu notation: This notation combines a spin up and a spin down fermionic fields oa a given band and of a given layer into a new Nambu spinor as

$$\psi_{l,i}=\begin{pmatrix}C_{l,i,\uparrow}\\C^\dagger_{l,i,\downarrow}\end{pmatrix}\quad\text{and}\quad\psi^\dagger_{l,i}=\begin{pmatrix}C^\dagger_{l,i,\uparrow} & C_{l,i,\downarrow}\end{pmatrix} \tag{14}$$

In term of these Nambu spinor, we can write the partition function as

$$Z=\int \mathrm{D}[\bar{\Delta},\Delta]\int \mathrm{D}[\psi^{\dagger},\psi]\exp\Big\{-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\int d^3r\Big[\bar{\Delta}(V^{-1})\Delta+\sum_{l,i}\psi^{\dagger}_{l,i}G^{-1}_{0,l,i}\psi_{l,i}$$
$$+\sum_{l,i,i'}\Big(\psi^{\dagger}_{l,i}\hat{T}^{i,i'}_{l,l+1}\psi_{l+1,i'}+\psi^{\dagger}_{l+1,i'}\hat{T}^{*i',i}_{l+1,l}\psi_{l,i}\Big)\Big]\Big\} \quad (15)$$

where

$$G^{-1}_{0,l,i}=\begin{pmatrix}\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\nabla^2-\mu_{\uparrow} & \Delta_{l,i}\\ \bar{\Delta}_{l,i} & \hbar\frac{\partial}{\partial\tau}+\frac{\hbar^2}{2m}\nabla^2+\mu_{\downarrow}\end{pmatrix} \quad (16)$$

$$\hat{T}^{i,i'}_{l,l+1}=\begin{pmatrix}T^{i,i'}_{l,l+1} & 0\\ 0 & -T^{i,i'}_{l,l+1}\end{pmatrix} \quad \text{and} \quad \hat{T}^{i',i}_{l+1,l}=\begin{pmatrix}T^{i',i}_{l+1,l} & 0\\ 0 & -T^{i',i}_{l+1,l}\end{pmatrix} \quad (17)$$

Introducing the phase factor: Here, all the fermionic fields and bosonic fields are complex. Hence they can be written in term of phase angle $\theta(\vec{r},\tau)$ as

$$\psi_{l,i}\rightarrow\begin{pmatrix}e^{i\theta_{l,i}/2} & 0\\ 0 & e^{-i\theta_{li}/2}\end{pmatrix}\psi_{l,i} \quad \text{and} \quad \Delta_{l,i}\rightarrow\Delta_{l,i}e^{i\theta_{l,i}} \quad (18)$$

Under these unitary transformations, the partition function becomes

$$Z=\int \mathrm{D}[\bar{\Delta},\Delta]\int \mathrm{D}[\psi^{\dagger},\psi]\exp\Big\{-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\int d^3r\Big[\sum_{l,l',i,i'}\bar{\Delta}_{l,i}(V^{-1})^{i,i'}_{l,l'}\Delta_{l',i'}e^{-i(\theta_{l,i}-\theta_{l',i'})}+\sum_{l,i}\psi^{\dagger}_{l,i}\Big(G^{-1}_{0,l,i}+F_{li}\Big)\psi_{l,i}$$
$$+\sum_{l,i,i'}\Big(\psi^{\dagger}_{l,i}\hat{T}^{i,i'}_{l,l+1}\psi_{l+1,i'}+\psi^{\dagger}_{l+1,i'}\hat{T}^{*i',i}_{l+1,l}\psi_{l,i}\Big)\Big]\Big\} \quad (19)$$

with

$$F_{li}=\left[\frac{i\hbar}{2}\frac{\partial\theta_{li}}{\partial\tau}+\frac{\hbar^2}{8m}(\nabla\theta_{li})^2\right]\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-\left[\frac{i\hbar^2}{2m}\nabla\theta_{li}\cdot\nabla+\frac{i\hbar^2}{4m}\nabla^2\theta_{li}\right]\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad (20)$$

and

$$\bar{\Delta}V^{-1}\Delta=\sum_{l,l',i,i'}\bar{\Delta}_{l,i}(V^{-1})^{i,i'}_{l,l'}\Delta_{l',i'}e^{-i(\theta_{l,i}-\theta_{l',i'})},\ \hat{T}^{i,i'}_{l,l+1}=\begin{pmatrix}T^{i,i'}_{l,l+1}e^{\frac{i}{2}(\theta_{l+1,i'}-\theta_{l,i})} & 0\\ 0 & -T^{i,i'}_{l,l+1}e^{-\frac{i}{2}(\theta_{l+1,i'}-\theta_{l,i})}\end{pmatrix} \quad (21)$$

Transformation to the reciprocal space

The goal of this section is to rewrite the partition function on in reciprocal space i.e. wave vector-frequency space. For this purpose, we use the Fourier transform for the fermionic fields $C_{l,i,\sigma}$ and as well as bosonic fields $\Delta_{l,i}$ as

$$C_{li\sigma}(\vec{r},\tau)=\frac{1}{\sqrt{\Omega}}\sum_{k,n}e^{-i\omega_n\tau+i\vec{k}\cdot\vec{r}}c_{li\sigma}(\vec{k},n) \quad \text{and} \quad C^{\dagger}_{li\sigma}(\vec{r},\tau)=\frac{1}{\sqrt{\Omega}}\sum_{k,n}e^{i\omega_n\tau-i\vec{k}\cdot\vec{r}}c^{\dagger}_{li\sigma}(\vec{k},n) \quad (22)$$

for fermionic fields and

$$\Delta_{l,i}(\vec{r},\tau)=\sum_{q,m}e^{-i\omega_m\tau+i\vec{q}\cdot\vec{r}}\Phi_{li}(\vec{q},m)\quad \text{and}\quad \bar{\Delta}_{li}(\vec{r},\tau)=\sum_{q,m}e^{i\omega_m\tau-i\vec{q}\cdot\vec{r}}\bar{\Phi}_{li}(\vec{q},m) \tag{23}$$

for bosonic fields. The Fourier transformed bosonic fields Φ_{li} still has the dimension of energy and the transformed fermionic fields c_{li} are dimensionless. where, ω_n is called the Matshubara frequency given by

$$\omega_n=\begin{cases}\dfrac{(2n+1)\pi}{\hbar\beta} & \text{for fermions}\\[2mm] \dfrac{2n\pi}{\hbar\beta} & \text{for bosons}\end{cases} \tag{24}$$

The phase factor θ_{li} and its derivatives do not take part in the transformation. Now, the partition function in reciprocal space is

$$\begin{aligned}Z=&\int \mathrm{D}[\bar{\Phi},\Phi]\int \mathrm{D}[\eta^\dagger,\eta]\exp\Big\{-\frac{1}{\hbar}\Big[\hbar\beta\Omega\sum_{q,m}\sum_{l,l',i,i'}\bar{\Phi}_{li}(\vec{q},m)(V^{-1})^{ii'}_{ll'}\Phi_{l'i'}(\vec{q},m)\times\\ &e^{-i(\theta_{li}-\theta_{l'i'})}+\hbar\sum_{k,n,k',n'}\sum_{l,i}\eta^\dagger_{li}(\vec{k},n)\bar{G}^{-1}_{0li}\eta_{li}(\vec{k}',n')+\hbar\sum_{k,n}\sum_{l,i}\eta^\dagger_{li}(\vec{k},n)\bar{F}_{li}\eta_{li}(\vec{k},n)\\ &+\hbar\sum_{k,n}\sum_{l,i,i'}\Big(\eta^\dagger_{li}(\vec{k},n)\bar{T}^{ii'}_{l,l+1}\eta_{l+1,i'}(\vec{k},n)+\eta^\dagger_{l+1,i'}(\vec{k},n)\bar{T}^{*i'i}_{l+1,l}\eta_{li}(\vec{k},n)\Big)\Big]\Big\}\end{aligned} \tag{25}$$

with the Nambu spinor in the reciprocal space as

$$\eta_{li}(\vec{k},n)=\begin{pmatrix}c_{li\uparrow}(\vec{k},n)\\ c^\dagger_{li\downarrow}(\vec{k},n)\end{pmatrix}\quad \text{and}\quad \eta^\dagger_{li}=\begin{pmatrix}c^\dagger_{li\uparrow}(\vec{k},n) & c_{li\downarrow}(\vec{k},n)\end{pmatrix} \tag{26}$$

and the inverse Green's function $\bar{G}^{-1}_{0li}$ is 2×2 matrix in reciprocal space and given by

$$\bar{G}^{-1}_{0li}=\begin{pmatrix}\left(-i\hbar\beta\omega_n+\beta\epsilon_k-\beta\mu_\uparrow\right)\delta(\vec{k}-\vec{k}')\delta_{nn'} & \beta\Phi_{li}(\vec{k}+\vec{k}',n+n')\\ \beta\bar{\Phi}_{li}(\vec{k}+\vec{k}',n+n') & \left(i\hbar\beta\omega_n-\beta\epsilon_k+\beta\mu_\downarrow\right)\delta(\vec{k}-\vec{k}')\delta nn'\end{pmatrix} \tag{27}$$

Where $\epsilon_k=\dfrac{\hbar^2k^2}{2m}$ is the free energy of a fermion. We are completely get rid of the operator version of G^{-1}_{0li}. Every element of this inverse Green's function is dimensionless. Again

$$\bar{F}_{li}=\left[\frac{i\hbar\beta}{2}\frac{\partial\theta_{li}}{\partial\tau}+\frac{\beta\hbar^2}{8m}(\nabla\theta_{li})^2+\frac{\beta\hbar^2}{2m}\vec{k}\cdot\nabla\theta_{li}\right]\begin{pmatrix}1&0\\0&-1\end{pmatrix}-\left[\frac{i\beta\hbar^2}{4m}\nabla^2\theta_{li}\right]\begin{pmatrix}1&0\\0&1\end{pmatrix} \tag{28}$$

and

$$\bar{T}^{i,i'}_{l,l+1} = \begin{pmatrix} \beta T^{i,i'}_{l,l+1} e^{\frac{i}{2}(\theta_{l+1,i'}-\theta_{l,i})} & 0 \\ 0 & -\beta T^{i,i'}_{l,l+1} e^{-\frac{i}{2}(\theta_{l+1,i'}-\theta_{l,i})} \end{pmatrix} \tag{29}$$

Saddle-point approximation: Here, the path-integral over the bosonic fields Φ_{li} is so difficult and almost impossible analytically. In order to tackle this difficulty, a simple approximation can be made in which all the bosonic fields $\bar{\Phi}_{li}(\vec{q},m)$ and $\Phi_{li}(\vec{q},m)$ are condensed in the state $\vec{q}=0$, $m=0$ state and the two pair fields as

$$\Phi_{li}(\vec{q},m) = \Delta_{0li}\delta(\vec{q})\delta_{m,0} \quad \text{and} \quad \bar{\Phi}_{li}(\vec{q},m) = \Delta^*_{0li}\delta(\vec{q})\delta_{m,0} \tag{30}$$

where Δ_{0li} still has the unit of energy and constant for the path-integral process. Using this approximation, the partition function is

$$Z = \int \mathrm{D}[\eta^\dagger,\eta]\exp\Big\{-\frac{1}{\hbar}\Big[\hbar\beta\Omega\sum_{l,l',i,i'}\Delta^*_{0li}(V^{-1})^{ii'}_{ll'}\Delta_{0l'i'}e^{-i(\theta_{li}-\theta_{l'i'})} + \hbar\sum_{k,n}\sum_{l,i}\eta^\dagger_{li}(\vec{k},n)\left(\bar{G}^{-1}_{0li}+\bar{F}_{li}\right)\eta_{li}(\vec{k},n)$$
$$+\hbar\sum_{k,n}\sum_{l,i,i'}\left(\eta^\dagger_{li}(\vec{k},n)\bar{T}^{ii'}_{l,l+1}\eta_{l+1,i'}(\vec{k},n)+\eta^\dagger_{l+1,i'}(\vec{k},n)\bar{T}^{*i'i}_{l+1,l}\eta_{li}(\vec{k},n)\right)\Big]\Big\} \tag{31}$$

with new inverse Green's function

$$\bar{G}^{-1}_{0li} = \begin{pmatrix} -i\hbar\beta\omega_n + \dfrac{\beta\hbar^2k^2}{2m} - \beta\mu_\uparrow & \beta\Delta_{0li} \\ \beta\Delta^*_{0li} & -i\hbar\beta\omega_n - \dfrac{\beta\hbar^2k^2}{2m} + \beta\mu_\downarrow \end{pmatrix} \tag{32}$$

Performing the Grassmann integral: The partition function Equation (31) can be written even simpler form as

$$Z = \int \mathrm{D}[\eta^\dagger,\eta]\exp\Big\{-\frac{1}{\hbar}\Big[\hbar\beta\Omega\sum_{l,l',i,i'}\Delta^*_{0li}(V^{-1})^{ii'}_{ll'}\Delta_{0l'i'}e^{-i(\theta_{li}-\theta_{l'i'})} + \hbar\sum_{k,n}\eta^\dagger(\vec{k},n)[\bar{G}^{-1}_0+\bar{F}+\bar{T}]\eta(\vec{k},)\Big]\Big\} \tag{33}$$

where, $\bar{G}^{-1}_0$, $\bar{F}$ and $\bar{T}$ are $N\times N$ (with N number of bands in the system) matrices the elements of which are $\bar{G}^{-1}_{0li}$, $\bar{F}_{li}$ and $\bar{T}^{ii'}_{l,l+1}$. After performing the path-integral over fermionic fields (Grassmann variables, $\eta^\dagger$ and η), we get

$$Z = \exp\Big\{-\frac{1}{\hbar}\Big[\hbar\beta\Omega\sum_{l,l',i,i'}\Delta^*_{0li}(V^{-1})^{ii'}_{ll'}\Delta_{0l'i'}e^{-i(\theta_{li}-\theta_{l'i'})} - \hbar\sum_{k,n}\ln(\det[\bar{G}^{-1}_0+\bar{F}+\bar{T}])\Big]\Big\} \tag{34}$$

Hence the action functional is

$$S = \hbar\beta\Omega\sum_{l,l',i,i'}\Delta^*_{0li}(V^{-1})^{ii'}_{ll'}\Delta_{0l'i'}e^{-i(\theta_{li}-\theta_{l'i'})} - \hbar\sum_{k,n}\mathrm{tr}_{l,i,\sigma}\ln[\bar{G}^{-1}_0+\bar{F}+\bar{T}] \tag{35}$$

Here the trace is taken over layer, band and spin indices. After investing some times for matrix manipulation and applying the Goldston mode, the action in Equation (35) takes the form

$$S=\hbar\beta\Omega\sum_{l,l',i,i'}\Delta^{*}_{0li}(V^{-1})^{ii'}_{ll'}\Delta_{0l'i'}e^{-i(\theta_{li}-\theta_{l'i'})}-\hbar\Big[\sum_{l,i}\left(-\frac{\beta\hbar^{2}\Omega N(0)}{4}\right)\left(\frac{\partial\theta_{li}}{\partial\tau}+\frac{e^{*}A_{l}^{0}}{\hbar}\right)^{2}$$
$$+\sum_{l,i}\left(-\frac{\beta\hbar^{2}\Omega N(0)\mu}{6m}\right)\left(\nabla\theta_{li}-\frac{e^{*}\vec{A}_{l}}{\hbar}\right)^{2}+\sum_{l,i,i'}\Big(\frac{2\beta T^{ii'}_{l,l+1}T^{i'i}_{l+1,l}\Omega N(0)}{\Delta^{2}_{0l+1,i'}-\Delta^{2}_{0li}}\ln\left(\frac{\Delta_{0l+1,i'}}{\Delta_{0li}}\right)\Delta_{0li}\Delta_{0l+1,i'}\cos(\theta_{l+1,i'}-\theta_{li})\qquad(36)$$
$$+2\beta\Omega N(0)\hbar\omega_{D}\zeta\delta_{ii'}+\beta\Omega N(0)\hbar^{2}\omega_{D}^{2}\delta_{ii'}\Big)\Big]$$

The corresponding Lagrangian density is given by

$$\mathrm{L}=\sum_{l,l',i,i'}\Delta^{*}_{0li}(V^{-1})^{ii'}_{ll'}\Delta_{0l'i'}e^{-i(\theta_{li}-\theta_{l'i'})}+\sum_{l,i}\left(\frac{\hbar^{2}N(0)}{4}\right)\left(\frac{\partial\theta_{li}}{\partial\tau}+\frac{e^{*}A_{l}^{0}}{\hbar}\right)^{2}+\sum_{l,i}\left(\frac{\hbar^{2}N(0)\mu}{6m}\right)\left(\nabla\theta_{li}-\frac{e^{*}\vec{A}_{l}}{\hbar}\right)^{2}$$
$$-\sum_{l,i,i'}\Big[\frac{2T^{ii'}_{l,l+1}T^{i'i}_{l+1,l}N(0)}{\Delta^{2}_{0l+1,i'}-\Delta^{2}_{0li}}\ln\left(\frac{\Delta_{0l+1,i'}}{\Delta_{0li}}\right)\Delta_{0li}\Delta_{0l+1,i'}\cos(\theta_{l+1,i'}-\theta_{li})+2N(0)\hbar\omega_{D}\zeta\delta_{ii'}+N(0)\hbar^{2}\omega_{D}^{2}\delta_{ii'}\Big]\qquad(37)$$

At low temperature, the chemical potential μ is equal to the Fermi energy i.e. $\mu=\epsilon_F$ and $\zeta=0$ since $\mu_\uparrow=\mu_\downarrow$. We also have,

$$N(0)=\frac{3n}{4\epsilon_F}=\frac{3}{4}\frac{k_F^2}{3\pi^2}\frac{2m}{\hbar^2k_F^2}=\frac{mk_F}{2\pi^2\hbar^2}$$

The effective Lagrangian is given by

$$\mathrm{L}_{\mathrm{eff}}=\sum_{l,i}\frac{\varepsilon_0}{2\lambda_{TF}^2}\left(\frac{\hbar}{e^*}\frac{\partial\theta_{li}}{\partial\tau}+A_l^0\right)^2+\sum_{l,i}\frac{\varepsilon_0c^2}{2\lambda_L^2}\left(\frac{\hbar}{e^*}\nabla\theta_{li}-\vec{A}_l\right)^2$$
$$+\sum_{l,l',i,i'}\Delta^{*}_{0li}(V^{-1})^{ii'}_{ll'}\Delta_{0l'i'}e^{-i(\theta_{li}-\theta_{l'i'})}-\sum_{l,i,i'}\Big[\frac{2T^{ii'}_{l,l+1}T^{i'i}_{l+1,l}N(0)}{\Delta^{2}_{0l+1,i'}-\Delta^{2}_{0li}}\ln\left(\frac{\Delta_{0l+1,i'}}{\Delta_{0li}}\right)\Delta_{0li}\Delta_{0l+1,i'}\cos(\theta_{l+1,i'}-\theta_{li})\qquad(38)$$
$$+N(0)\hbar^2\omega_D^2\delta_{ii'}\Big]+\sum_{l}\left[\frac{\varepsilon_0}{2}E^2_{l,l+1}+\frac{\varepsilon_0c^2}{2}B^2_{l,l+1}\right]$$

where, n is the concentration of electronic charge, k_F is the Fermi wave vector, $\lambda_{TF}=\sqrt{\frac{\varepsilon_0\pi^2\hbar^2}{e^2mk_F}}$ is the Tomas Fermi charge screening length and $\lambda_L=\sqrt{\frac{\varepsilon_0mc^2}{ne^2}}$ is the London penetration depth, $\vec{E}_{l,l+1}$ and $\vec{B}_{l,l+1}$ are electric and magnetic fields between layer l and $l+1$.

Application to the long Josephson junction: Consider the stack of long Josephson junction with length along x-direction and junction system along z-direction. External magnetic fields are applied along the y-direction, which introduce the homogeneous phase difference along the x-direction. The system is assumed to uniform along the y-direction and the problem becomes two dimensional. The system is biased with an external potential difference across the junction i.e. the electric field is along z-direction. Now the Lagrangian density in two dimensional system becomes.

$$\mathrm{L}_{\mathrm{eff}}=\frac{\varepsilon_0 d}{2\lambda_{TF}^2}\sum_{l,i}\left(\frac{\hbar}{e^*}\frac{\partial\theta_l^i}{\partial\tau}+A_l^0\right)^2+\frac{\varepsilon_0 c^2 d}{2\lambda_L^2}\sum_{l,i}\left(\frac{\hbar}{e^*}\frac{\partial\theta_l^i}{\partial x}-A_l^x\right)^2+\sum_{l,i,i'}\frac{\hbar}{e^*}J_{ll}^{ii'}\cos(\theta_{li}-\theta_{li'})$$
$$-\sum_{l,i,i'}\left[\frac{\hbar}{e^*}j_{l,l+1}^{ii'}\cos(\theta_{l+1,i'}-\theta_{li})+N(0)\hbar^2\omega_D^2\delta_{ii'}\right]+\sum_l\left[\frac{\varepsilon_0 b}{2}(E_l^z)^2+\frac{\varepsilon_0 c^2 b}{2}(B_l^y)^2\right] \tag{39}$$

where d is the thickness of the superconducting layer and b is the thickness of junction material. The inter-band Josephson coupling constant is

$$J_{ll}^{ii'}=\frac{e^*}{\hbar}\Delta_{0li}^*(V^{-1})_{ll}^{ii'}\Delta_{0li'} \tag{40}$$

and Josephson tunneling coupling constant

$$j_{l,l+1}^{ii'}=\frac{e^*}{\hbar}\frac{2T_{l,l+1}^{ii'}T_{l+1,l}^{i'i}N(0)}{\Delta_{0l+1,i'}^2-\Delta_{0li}^2}\ln\left(\frac{\Delta_{0l+1,i'}}{\Delta_{0li}}\right)\Delta_{0li}\Delta_{0l+1,i'} \tag{41}$$

The z-component of electric field in between l^{th} and $(l+l)^{\text{th}}$ layer is
\

$$E_{l,l+1}^z=-\frac{\partial A_{l,l+1}^z}{\partial t}-\frac{1}{b}\left(A_{l+1}^0-A_l^0\right) \tag{42}$$

and the y-component of magnetic field in between l^{th} and $(l+l)^{\text{th}}$ layer is

$$B_{l,l+1}^y=\frac{1}{b}\left(A_{l+1}^x-A_l^x\right)-\frac{\partial A_{l,l+1}^z}{\partial x} \tag{43}$$

with

$$A_{l,l+1}^z=\frac{1}{b}\int_{-b/2}^{+b/2}A^z(z)dz \tag{44}$$

We can introduce the gauge invariant phase difference $\varphi_{l,l+1}^{ii'}$ as

$$\varphi_{l,l+1}^{ii'}=\theta_{l+1}^{i'}-\theta_l^i-\frac{be^*}{\hbar}A_{l,l+1}^z \tag{45}$$

Then we can have

$$\cos\left(\theta_{l+1}^{i'}-\theta_l^i\right)=\cos\left(\varphi_{l,l+1}^{ii'}+\frac{be^*}{\hbar}A_{l,l+1}^z\right)=\cos\varphi_{l,l+1}^{ii'}$$

and $\theta_l^{i'}-\theta_l^i=\chi_{ll}^{ii'}$ is the intra-layer inter-band phase difference.

Under these conditions, the Lagrangian density equation (39) can be minimized using the Euler-Lagrange equation. Applying the Euler-Lagrange equation with respect to A_k^0, A_{k+1}^0, A_k^x, A_{k+1}^x, $\frac{\partial A_{k,k+1}^z}{\partial x}$, θ_k^j with k as new layer index and j and new band index, then simplifying the system of equations, we get

$$\frac{\partial^2\chi_{kk}^{j'j}}{\partial\bar{t}^2}-\frac{\partial^2\chi_{kk}^{j'j}}{\partial\bar{x}^2}-\sum_i\left(\frac{J_{kk}^{ij'}}{J_0}\sin\chi_{kk}^{ij'}-\frac{J_{kk}^{ij}}{J_0}\sin\chi_{kk}^{ij}\right)+\sum_{i'}\left(\frac{J_{kk}^{j'i'}}{J_0}\sin\chi_{kk}^{j'i'}-\frac{J_{kk}^{ji'}}{J_0}\sin\chi_{kk}^{ji'}\right)$$
$$+\sum_i\left(\frac{j_{k-1,k}^{ij'}}{J_0}\sin\varphi_{k-1,k}^{ij'}-\frac{j_{k-1,k}^{ij}}{J_0}\sin\varphi_{k-1,k}^{ij}\right)-\sum_{i'}\left(\frac{j_{k,k+1}^{j'i'}}{J_0}\sin\varphi_{k,k+1}^{j'i'}-\frac{j_{k,k+1}^{ji'}}{J_0}\sin\varphi_{k,k+1}^{ji'}\right)=0 \tag{46}$$

$$\frac{\partial^2\varphi_{k,k+1}^{jj'}}{\partial\bar{t}^2}-\frac{\partial^2\varphi_{k,k+1}^{jj'}}{\partial\bar{x}^2}-\frac{1}{N_{b,k}N_{b,k+1}}\sum_{i,i'}\frac{\partial^2\varphi_{k,k+1}^{ii'}}{\partial\bar{t}^2}+\frac{1}{N_{b,k}N_{b,k+1}}\sum_{i,i'}\frac{\partial^2\varphi_{k,k+1}^{ii'}}{\partial\bar{x}^2}$$
$$+\frac{1}{2J_0}\sum_{i,i'}\left[\frac{1}{N_{b,k}}j_{k-1,k}^{ii'}\sin\varphi_{k-1,k}^{ii'}+\frac{1}{N_{b,k+1}}j_{k+1,k+2}^{ii'}\sin\varphi_{k+1,k+2}^{ii'}+\left(\frac{db}{\lambda_{TF}^2}-\frac{db}{\lambda_L^2}\right)j_{k,k+1}^{ii'}\sin\varphi_{k,k+1}^{ii'}\right]$$
$$-\sum_i\left(\frac{J_{k+1,k+1}^{ij'}}{J_0}\sin\chi_{k+1,k+1}^{ij'}-\frac{J_{kk}^{ij}}{J_0}\sin\chi_{kk}^{ij}\right)+\sum_{i'}\left(\frac{J_{k+1,k+1}^{j'i'}}{J_0}\sin\chi_{k+1,k+1}^{j'i'}-\frac{J_{kk}^{ji'}}{J_0}\sin\chi_{kk}^{ji'}\right)$$
$$+\sum_i\left(\frac{j_{k,k+1}^{ij'}}{J_0}\sin\varphi_{k,k+1}^{ij'}-\frac{j_{k-1,k}^{ij}}{J_0}\sin\varphi_{k-1,k}^{ij}\right)-\sum_{i'}\left(\frac{j_{k+1,k+2}^{j'i'}}{J_0}\sin\varphi_{k+1,k+2}^{j'i'}-\frac{j_{k,k+1}^{ji'}}{J_0}\sin\varphi_{k,k+1}^{ji'}\right)=0 \tag{47}$$

where, $N_{b,k}$ and $N_{b,k+1}$ are the number of bands in k^{th} and $(k+1)^{\text{th}}$ layers respectively, $\overline{t}=\frac{ct}{\lambda_L}$ and $\overline{x}=\frac{x}{\lambda_{TF}}$.

Equations (46) and (47) jointly describe the complete phase dynamics of stack of long Josephson junction of multi-band superconductors.

RESULTS AND DISCUSSION: Consider a two-gap superconductor like MgB_2, which contains two bands s and d act as two channels for condensates. Now the band index is $i=s,d$. Equations (46) and (47) becomes

$$\frac{\partial^2\chi_{kk}^{sd}}{\partial\overline{t}^2}-\frac{\partial^2\chi_{kk}^{sd}}{\partial\overline{x}^2}+\frac{4J_{kk}^{sd}}{J_0}\sin\chi_{kk}^{sd}+\left(\frac{j_{k-1,k}^{ss}}{J_0}\sin\varphi_{k-1,k}^{ss}-\frac{j_{k-1,k}^{sd}}{J_0}\sin\varphi_{k-1,k}^{sd}\right)+\left(\frac{j_{k-1,k}^{ds}}{J_0}\sin\varphi_{k-1,k}^{ds}-\frac{j_{k-1,k}^{dd}}{J_0}\sin\varphi_{k-1,k}^{dd}\right)$$
$$-\left(\frac{j_{k,k+1}^{ss}}{J_0}\sin\varphi_{k,k+1}^{ss}-\frac{j_{k,k+1}^{ds}}{J_0}\sin\varphi_{k,k+1}^{ds}\right)-\left(\frac{j_{k,k+1}^{sd}}{J_0}\sin\varphi_{k,k+1}^{sd}-\frac{j_{k,k+1}^{dd}}{J_0}\sin\varphi_{k,k+1}^{dd}\right)=0 \qquad (48)$$

and

$$\frac{\partial^2\varphi_{k,k+1}^{ss}}{\partial\overline{t}^2}-\frac{\partial\varphi_{k,k+1}^{ss}}{\partial\overline{x}^2}-\frac{1}{4}\sum_{\substack{i=s,d\\i'=s,d}}\frac{\partial^2\varphi_{k,k+1}^{ii'}}{\partial\overline{t}^2}+\frac{1}{4}\sum_{\substack{i=s,d\\i'=s,d}}\frac{\partial^2\varphi_{k,k+1}^{ii'}}{\partial\overline{x}^2}$$
$$+\frac{1}{2J_0}\sum_{\substack{i=s,d\\i'=s,d}}\left[\frac{1}{2}j_{k-1,k}^{ii'}\sin\varphi_{k-1,k}^{ii'}+\frac{1}{2}j_{k+1,k+2}^{ii'}\sin\varphi_{k+1,k+2}^{ii'}+\left(\frac{db}{\lambda_{TF}^2}-\frac{db}{\lambda_L^2}\right)j_{k,k+1}^{ii'}\sin\varphi_{k,k+1}^{ii'}\right]$$
$$-\sum_{i=s,d}\left(\frac{J_{k+1,k+1}^{is}}{J_0}\sin\chi_{k+1,k+1}^{is}-\frac{J_{kk}^{is}}{J_0}\sin\chi_{kk}^{is}\right)+\sum_{i'=s,d}\left(\frac{J_{k+1,k+1}^{si'}}{J_0}\sin\chi_{k+1,k+1}^{si'}-\frac{J_{kk}^{si'}}{J_0}\sin\chi_{kk}^{si'}\right)$$
$$+\sum_{i=s,d}\left(\frac{j_{k,k+1}^{is}}{J_0}\sin\varphi_{k,k+1}^{is}-\frac{j_{k-1,k}^{is}}{J_0}\sin\varphi_{k-1,k}^{is}\right)-\sum_{i'=s,d}\left(\frac{j_{k+1,k+2}^{si'}}{J_0}\sin\varphi_{k+1,k+2}^{si'}-\frac{j_{k,k+1}^{si'}}{J_0}\sin\varphi_{k,k+1}^{si'}\right)=0 \qquad (49)$$

and

$$\frac{\partial^2\varphi_{k,k+1}^{dd}}{\partial\overline{t}^2}-\frac{\partial\varphi_{k,k+1}^{dd}}{\partial\overline{x}^2}-\frac{1}{{}^24}\sum_{\substack{i=s,d\\i'=s,d}}\frac{\partial^2\varphi_{k,k+1}^{dd}}{\partial\overline{t}^2}+\frac{1}{4}\sum_{\substack{i=s,d\\i'=s,d}}\frac{\partial\varphi_{k,k+1}^{dd}}{\partial\overline{x}^2}$$
$$+\frac{1}{2J_0}\sum_{\substack{i=s,d\\i'=s,d}}\left[\frac{1}{2}j_{k-1,k}^{ii'}\sin\varphi_{k-1,k}^{ii'}+\frac{1}{2}j_{k+1,k+2}^{ii'}\sin\varphi_{k+1,k+2}^{ii'}+\left(\frac{db}{\lambda_{TF}^2}-\frac{db}{\lambda_L^2}\right)j_{k,k+1}^{ii'}\sin\varphi_{k,k+1}^{ii'}\right]$$
$$-\sum_{i=s,d}\left(\frac{J_{k+1,k+1}^{id}}{J_0}\sin\chi_{k+1,k+1}^{id}-\frac{J_{kk}^{id}}{J_0}\sin\chi_{kk}^{id}\right)+\sum_{i'=s,d}\left(\frac{J_{k+1,k+1}^{di'}}{J_0}\sin\chi_{k+1,k+1}^{di'}-\frac{J_{kk}^{di'}}{J_0}\sin\chi_{kk}^{di'}\right)$$
$$+\sum_{i=s,d}\left(\frac{j_{k,k+1}^{id}}{J_0}\sin\varphi_{k,k+1}^{id}-\frac{j_{k-1,k}^{id}}{J_0}\sin\varphi_{k-1,k}^{id}\right)-\sum_{i'=s,d}\left(\frac{j_{k+1,k+2}^{di'}}{J_0}\sin\varphi_{k+1,k+2}^{di'}-\frac{j_{k,k+1}^{di'}}{J_0}\sin\varphi_{k,k+1}^{di'}\right)=0 \qquad (50)$$

Since s and d bands are identical in all the layers, we can assume $\chi_{kk}^{sd} = \chi$ and $J_{kk}^{sd} = J$. Thus We have, $\varphi_{k,k+1}^{sd} + \varphi_{k,k+1}^{ds} = \varphi_{k,k+1}^{ss} + \varphi_{k,k+1}^{dd}$ with $\varphi_{k,k+1}^{sd} = \varphi_{k,k+1}^{dd} + \chi$ and $\varphi_{k,k+1}^{ds} = \varphi_{k,k+1}^{ss} - \chi$. Hence the equations (48), (49) and (50) for a typical LJJ with single layer barrier becomes.

$$\frac{\partial^2 \chi}{\partial \bar{t}^2} - \frac{\partial^2 \chi}{\partial \bar{x}^2} + \frac{4J}{J_0}\sin\chi - \frac{j_{12}^{ss}}{J_0}\sin\varphi_{12}^{ss} + \frac{j_{12}^{ds}}{J_0}\sin(\varphi_{12}^{ss} - \chi) - \frac{j_{12}^{sd}}{J_0}\sin(\varphi_{12}^{dd} + \chi) + \frac{j_{k,k+1}^{dd}}{J_0}\sin\varphi_{12}^{dd} = 0 \qquad (51)$$

$$\begin{aligned} &\frac{\partial^2 \varphi_{12}^{ss}}{\partial \bar{t}^2} - \frac{\partial^2 \varphi_{12}^{dd}}{\partial \bar{t}^2} - \frac{\partial \varphi_{12}^{ss}}{\partial \bar{x}^2} + \frac{\partial \varphi_{12}^{dd}}{\partial \bar{x}^2} + \frac{db}{J_0}\left(\frac{1}{\lambda_{TF}^2} - \frac{1}{\lambda_L^2}\right)\left(j_{12}^{ss}\sin\varphi_{12}^{ss} + j_{12}^{sd}\sin(\varphi_{12}^{dd} + \chi) + j_{12}^{ds}\sin(\varphi_{12}^{ss} - \chi) + j_{12}^{dd}\sin\varphi_{12}^{dd} \right) \\ &+ \frac{4 j_{12}^{ss}}{J_0}\sin\varphi_{12}^{ss} + \frac{2 j_{12}^{ds}}{J_0}\sin(\varphi_{12}^{ss} - \chi) + \frac{2 j_{12}^{sd}}{J_0}\sin(\varphi_{12}^{dd} + \chi) = 0 \end{aligned} \qquad (52)$$

$$\begin{aligned} &\frac{\partial^2 \varphi_{12}^{dd}}{\partial \bar{t}^2} - \frac{\partial^2 \varphi_{12}^{ss}}{\partial \bar{t}^2} - \frac{\partial^2 \varphi_{12}^{dd}}{\partial \bar{x}^2} + \frac{\partial^2 \varphi_{12}^{ss}}{\partial \bar{x}^2} + \frac{db}{J_0}\left(\frac{1}{\lambda_{TF}^2} - \frac{1}{\lambda_L^2}\right)\left(j_{12}^{ss}\sin\varphi_{12}^{ss} + j_{12}^{sd}\sin(\varphi_{12}^{dd} + \chi) + j_{12}^{ds}\sin(\varphi_{12}^{ss} - \chi) + j_{12}^{dd}\sin\varphi_{12}^{dd} \right) \\ &+ \frac{4 j_{12}^{dd}}{J_0}\sin\varphi_{12}^{dd} + \frac{2 j_{12}^{ds}}{J_0}\sin(\varphi_{12}^{ss} - \chi) + \frac{2 j_{12}^{sd}}{J_0}\sin(\varphi_{12}^{dd} + \chi) = 0 \end{aligned} \qquad (53)$$

Therefore, equations (51), (52) and (53) completely describe the phase dynamics in LJJ of two-gap superconductor like MgB_2 for single layer barrier. These equations almost identical to those derived by Kim and Ghimire \cite{kim2012} with some perturbation term which they have not considered. Similarly we can apply our generalized equations of phase dynamic i.e. equations (46) and (47) can be applied for higher band and multi-layered LJJ.

CONCLUSIONS: As discussed in the above section, it is predicted that, our generalized equations for phase dynamics can describe the complete phase dynamic in the system of LJJ. These equations are called perturbed sine-Gordon equations. The analytical solutions of this system of equations are impossible. They can be solved numerically imposing some appropriate boundary condition and initial profile of phases. The appropriate boundary conditions are the external magnetic field and biasing voltage. After solving this system of equations, we can obtained intra-band phase difference χ (assume same for all layers) and inter-band phase difference φ (which are different for each interplay). Knowing these parameter, many phenomena in the system of LJJ can be explained such as collective oscillation, THz frequency emission, TRS breaking etc.

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